THE

MATHEMATICAL GAZETTE.

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WITH THE CO-OPERATION OF

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SOLUTIONS.

683. [K. 10. e.] *On the sides BC, CA, AB of a given triangle are taken points P, Q, R such that the triangle PQR is of given species. Prove that the locus of the circumcentre of PQR is a straight line.*

The circles AQR, BRP, CPQ will intersect in a fixed point O.

For $BOC - BAC = OBA + OCA = OPR + OPQ$.

Hence $BOC = A + P$, and is $\therefore$ given. Similarly $COA = B + Q$.

Now let S be the circumcentre of PQR. Then since $OPR = OBR$ and $ORP = OBP$; $\therefore$ the $\triangle OPR$ is of given species; $\therefore$ $OP : PR$ is constant. Also since PQR is of given species; $\therefore$ $SP : PR$ is constant; $\therefore$ $OP : PS$ is constant. Also $\angle OPS = SPR - OPR$ and is $\therefore$ given.

Hence the $\triangle OPS$ is of given species, and one vertex O is fixed, while a second P moves along a given straight line, viz. BC. Hence the third vertex O also describes a straight line.

684. [J. 1. 8.] *If h_n be the sum of all the homogeneous products of n dimensions wherein no letter is to appear raised to so high a power as m, prove that*

$$h_n = H_n - (\Sigma a_1{}^m) H_{n-m} + (\Sigma a_1{}^m a_2{}^m) H_{n-2m} - (\Sigma a_1{}^m a_2{}^m a_3{}^m) H_{n-3m} + \dots,$$

where H_n is the sum of all the homogeneous products of n dimensions.

The sum of the homogeneous products required is the coefficient of x^n in

$$(1 + a_1 x + \dots + a_1{}^{m-1} x^{m-1})(1 + a_2 x + \dots + a_2{}^{m-1} x^{m-1}) \dots$$
$$= \frac{1 - a_1{}^m x^m}{1 - a_1 x} \cdot \frac{1 - a_2{}^m x^m}{1 - a_2 x} \dots .$$

But
$$\frac{1}{(1 - a_1 x)(1 - a_2 x) \dots} = 1 + \Sigma H_n x^n.$$

Hence the above expression is

$$(1 - \Sigma a_1{}^m x^m + \Sigma a_1{}^m a_2{}^m x^{2m} - \dots)(1 + \Sigma H_n x^n),$$

and the coefficient of x^n in this is

$$H_n - (\Sigma a_1{}^m) H_{n-m} + (\Sigma a_1{}^m a_2{}^m) H_{n-2m} - \dots .$$

685. [L[1]. 5. b.] *From any point on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *three normals other than the one at the point are drawn. Shew that the centre of the circle through their feet lies on the conic*

$$a^6x^2+b^6y^2=\tfrac{1}{4}a^4b^4.$$

Suppose the circle through α, β, γ cuts the curve again in δ', and the normals at α, β, γ meet in δ. Then $\alpha+\beta+\gamma+\delta'$ is an even multiple of π, and $\alpha+\beta+\gamma+\delta$ an odd multiple; $\therefore\ \delta'-\delta$ is an odd multiple. The centre of the circle is

$$X=\frac{a^2-b^2}{4a}(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta')=\frac{a^2-b^2}{4a}(\cos\alpha+\cos\beta+\cos\gamma-\cos\delta),$$

$$Y=\frac{b^2-a^2}{4b}(\sin\alpha+\sin\beta+\sin\gamma+\sin\delta')=\frac{b^2-a^2}{4b}(\sin\alpha+\sin\beta+\sin\gamma-\sin\delta).$$

Now if the normal at θ passes through δ, we have

$$a^2\sin\theta\cos\delta-b^2\cos\theta\sin\delta=(a^2-b^2)\sin\theta\cos\theta.$$

Putting $\cos\theta=\lambda$, this equation is

$$\{a^2\cos\delta-(a^2-b^2)\lambda\}^2(1-\lambda^2)=b^4\sin^2\delta\,.\,\lambda^2,$$

or $$(a^2-b^2)^2\lambda^4-2a^2(a^2-b^2)\cos\delta\,.\,\lambda^3+\ldots=0\,;$$

$$\therefore\ \Sigma\cos\alpha=\frac{2a^2}{a^2-b^2}\cos\delta\,;$$

$$\therefore\ \cos\alpha+\cos\beta+\cos\gamma-\cos\delta=\frac{2b^2}{a^2-b^2}\cdot\cos\delta\,;$$

$$\therefore\ X=\frac{b^2}{2a}\cos\delta,\ \ i.e.\ \ a^3X=\frac{a^2b^2}{2}\cos\delta,$$

Similarly, $b^3Y=\frac{a^2b^2}{2}\sin\delta$. Hence the given locus.

686. [L[1]. 4. c.] *ABC is a triangle and conics are drawn touching AB at B and AC at C. Tangents are drawn to these conics parallel to a given straight line. Shew that the locus of their points of contact is a conic circumscribing the triangle ABC.*

Any one of the conics is of the form $\beta\gamma=k\alpha^2$, and the tangent at (α',β',γ') is

$$2k\alpha'\alpha-\gamma'\beta-\beta'\gamma=0\quad\text{or}\quad 2\beta'\gamma'\alpha-\gamma'\alpha'\beta-\alpha'\beta'\gamma=0.$$

If this is $\parallel$ to $l\alpha+m\beta+n\gamma=0$, we have

$$\begin{vmatrix} 2\beta'\gamma', & -\gamma'\alpha', & -\alpha'\beta' \\ l, & m, & n \\ a, & b, & c \end{vmatrix}=0,$$

whence the locus of (α',β',γ') is the conic

$$2(mc-nb)\beta\gamma+(lc-na)\gamma\alpha+(ma-lb)\alpha\beta=0.$$

687. [R. 7. a. β.] *A particle of mass m slides down a smooth inclined plane of mass M and angle α. The wedge can slide on a smooth horizontal plane. If h be the initial height of the particle above the plane, shew that it will reach the horizontal plane in a time t given by*

$$t^2=\frac{2h}{g}\left(1+\frac{M}{M+m}\cot^2\alpha\right),$$

and that in this time the wedge will have moved a horizontal distance

$$\frac{mh}{M+m}\cot\alpha.$$

Let f_1, f_2 be the accelerations of m along and perpendicular to the plane, f that of the wedge horizontally, R the pressure. Then

$$f_1 = g\sin\alpha, \quad mf_2 = mg\cos\alpha - R,$$
$$Mf = R\sin\alpha, \quad f_2 = f\sin\alpha,$$

whence $$R = g\cos\alpha \Big/ \left(\frac{1}{m} + \frac{\sin^2\alpha}{M}\right) \text{ and } f = \frac{mg\cot\alpha}{M\operatorname{cosec}^2\alpha + m}.$$

The vertical acceleration is

$$F = f_1\sin\alpha + f_2\cos\alpha = g - \frac{R}{m}\cos\alpha = \frac{M+m}{M\operatorname{cosec}^2\alpha + m}\cdot g.$$

and the time is given by $h = \frac{1}{2}Ft^2$, *i.e.* $t^2 = \frac{2h}{F}$, as required.

If in this time the wedge moves a distance x, then $x = \frac{1}{2}ft^2$;

$$\therefore \frac{x}{h} = \frac{f}{F} = \frac{m\cot\alpha}{M+m}.$$

688. [R. 7. b. γ.] *A ball is projected from a point in a smooth plane inclined at an angle α to the horizon in a direction making an angle β with the plane, and in the vertical plane through the plane's normal. If the coefficient of elasticity be e, prove that the condition that the ball should return to the point of projection is that*

$$\log\{1 - (1-e)\cot\alpha\cot\beta\}/\log e$$

should be a positive integer.

At each impact the velocity perpendicular to the plane is diminished in the ratio $e:1$. Hence the times in successive trajectories is

$$\frac{2u\sin\beta}{g\cos\alpha}, \quad \frac{2eu\sin\beta}{g\cos\alpha}, \quad \text{etc.,}$$

so that the time of the first n trajectories is $\frac{2u\sin\beta}{g\cos\alpha}\cdot\frac{1-e^n}{1-e}(=t)$.

If in this time the particle has returned to the point of projection

$$u\cos\beta \,.\, t - \tfrac{1}{2}g\sin\alpha \,.\, t^2 = 0; \quad \therefore t = \frac{2u\cos\beta}{g\sin\alpha}.$$

Equating the values of t, we get $e^n = 1 - (1-e)\cot\alpha\cot\beta$, whence, taking logarithms, the result follows.

689. [K. 7. d.] *Shew that if each of two pairs of opposite vertices of a quadrilateral is conjugate with regard to a circle, the third pair is also, and that the circle is one of a coaxal system of which the line of collinearity of the middle points of the diagonals is the radical axis.*

Let P and Q be any two points conjugate for a circle, centre O. Then, if QY be drawn perpendicular to OP, QY will be the polar of P; $\therefore OY.OP = r^2$, where r is the radius. But Y lies on the circle on PQ as diameter; $\therefore$ the tangent to this circle from $O = r$, *i.e.* the two circles cut orthogonally.

Now the circles described on the diagonals of a complete quadrilateral are coaxal, and by hypothesis two of them cut a given circle orthogonally; $\therefore$ so also does the third. Hence, conversely, the two remaining vertices are conjugate for the circle. Further, since the given circle cuts a coaxal system (viz., the circles on the diagonals) orthogonally, it belongs to a coaxal system whose radical axis is the line of centres of the other system, viz., the line through the middle points of the diagonals.

690. [B. 1. a.] *Prove that the value of the determinant*

$$\begin{vmatrix} 1+x^2, & x, & 0, & 0, & \dots \\ x, & 1+x^2, & x, & 0, & \dots \\ 0, & x, & 1+x^2, & x, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$

of the m^{th} order is $1+x^2+x^4+\dots+x^{2m}$.

Calling the given determinant Δ_m, it is evident on expanding that

$$\Delta_m=(1+x^2)\Delta_{m-1}-x^2\Delta_{m-2},$$

$$i.e.\quad \Delta_m-\Delta_{m-1}=x^2(\Delta_{m-1}-\Delta_{m-2}).$$

Similarly $\quad \Delta_{m-1}-\Delta_{m-2}=x^2(\Delta_{m-2}-\Delta_{m-3}),$

$$\dots\dots\dots\dots\dots\dots$$

$$\Delta_3-\Delta_2=x^2(\Delta_2-\Delta_1).$$

Hence, multiplying, $\Delta_m-\Delta_{m-1}=x^{2m-4}(\Delta_2-\Delta_1)=x^{2m}$;

$$\therefore\ \Delta_m-\Delta_1=x^4+x^6+\dots+x^{2m},$$

and $\Delta_1=1+x^2$.

691. [K. 20. c.] *If* $\alpha=\dfrac{2\pi}{13}$ *and* $\beta=\dfrac{\pi}{9}$, *shew that*

$$(\cos\alpha+\cos 5\alpha)(\cos 2\alpha+\cos 3\alpha)(\cos 4\alpha+\cos 6\alpha)$$
$$=-\tfrac{1}{8}=-2\cos\beta\cos 2\beta\cos 3\beta\cos 4\beta.$$

Since $\cos\alpha+\cos 5\alpha=2\cos 2\alpha\cos 3\alpha$, etc., the first expression is

$$-8.\prod_{r=1}^{r=6}\cos\frac{r\pi}{13},\quad\text{since}\quad \cos\frac{10\pi}{13}=-\cos\frac{3\pi}{13}.$$

If in the equation $\qquad \dfrac{x^{13}-1}{x-1}=\displaystyle\prod_{r=1}^{r=6}\left(x^2-2x\cos\frac{2r\pi}{13}+1\right),$

we put $x=-1$, we get

$$1=2^{12}.\prod_{r=1}^{r=6}\cos^2\frac{r\pi}{13}.$$

On taking the square root, the positive sign must be taken. Hence $\displaystyle\prod_{r=1}^{r=6}\cos\frac{r\pi}{13}=\frac{1}{2^6}$, and the first result follows.

Similarly the second result may be obtained from the equation

$$\frac{x^9-1}{x-1}=\prod_{r=1}^{r=4}\left(x^2-2x\cos\frac{2r\pi}{9}+1\right).$$

692. [D. 2. b. β.] *Shew that*

$$\frac{\sin\theta}{1-2ax+a^2x^2\sec^2\theta}=\sum_0^\infty a^n x^n\sec^n\theta\sin(n+1)\theta.$$

Since $1-2ax+a^2x^2\sec^2\theta=(1-ax\sec\theta.e^{i\theta})(1-ax\sec\theta.e^{-i\theta})$, we have

$$\frac{\sin\theta}{1-2ax+a^2x^2\sec^2\theta}=\frac{1}{2i}\left(\frac{e^{i\theta}}{1-ax\sec\theta.e^{i\theta}}-\frac{e^{-i\theta}}{1-ax\sec\theta.e^{-i\theta}}\right)$$

$$=\frac{1}{2i}\left(e^{i\theta}.\sum_0^\infty a^n x^n\sec^n\theta.e^{ni\theta}-e^{-i\theta}.\sum_0^\infty a^n x^n\sec^n\theta.e^{-ni\theta}\right)$$

$$=\frac{1}{2i}.\sum_0^\infty a^n x^n\sec^n\theta\left(e^{\overline{n+1}\,i\theta}-e^{-\overline{n+1}\,i\theta}\right)$$

$$=\sum_0^\infty a^n x^n\sec^n\theta\sin(n+1)\theta.$$

693. [L[1]. 16. b.] *A triangle is inscribed in the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *and has its centre of gravity at the centre of the ellipse. Shew that the locus of the circumcentre is*

$$a^2x^2+b^2y^2=\{\tfrac{1}{4}(a^2-b^2)\}^2.$$

The co-ordinates of the centre of the circle through α, β, γ are

$$X=\frac{a^2-b^2}{4a}\{\Sigma\cos\alpha+\cos(\alpha+\beta+\gamma)\},$$

$$Y=-\frac{a^2-b^2}{4b}\{\Sigma\sin\alpha-\sin(\alpha+\beta+\gamma)\}.$$

But since the centroid is at the centre of the ellipse

$$\therefore\ \Sigma\cos\alpha=\Sigma\sin\alpha=0,$$

whence $$aX=\tfrac{1}{4}(a^2-b^2)\cos(\alpha+\beta+\gamma),$$

$$bY=\tfrac{1}{4}(a^2-b^2)\sin(\alpha+\beta+\gamma),$$

and the locus of (X, Y) is as stated.

694. [L[1]. 4. c.] *Tangents are drawn from a point O to an ellipse so as to intercept a fixed length on the tangent at a given point P. Shew that the locus of O is a conic which has four-point contact with the ellipse at the other extremity P′ of the diameter through P.*

Take the diameter through P and the tangent at P as axes. Then if $CP=a$, and the semi-conjugate diameter $=b$, the equation to the ellipse is

$$\frac{(x+a)^2}{a^2}+\frac{y^2}{b^2}=1,\ \text{ or }\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2x}{a}=0.$$

Let O be (x', y'). Then the tangents from O meet the axis of y where

$$\frac{y^2}{b^2}\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{2x'}{a}\right)=\left(\frac{yy'}{b^2}+\frac{x'}{a}\right)^2,\ \text{ or }\ \frac{y^2}{b^2}\left(\frac{x'}{a}+2\right)-\frac{2yy'}{b^2}-\frac{x'}{a}=0.$$

If the difference of the values of y given by this equation is constant and equal to k, we find

$$\frac{y'^2}{b^2}+\frac{x'^2}{a^2}+\frac{2x'}{a}=\frac{k^2}{b^2}\left(\frac{x'}{a}+2\right)^2,$$

shewing that the locus of (x', y') is a conic having four-point contact with the given conic, the common tangent being $\frac{x}{a}+2=0$, *i.e.* the tangent at P'.

695. [R. 1. d.] *If U and u be the velocities of two particles whose weights are W and w, and if ρ be their relative velocity, V the velocity of their centre of inertia, and θ the angle between V and ρ, prove that*

$$U^2=V^2+\left(\frac{w}{W+w}.\rho\right)^2+\frac{2w}{W+w}.V\rho\cos\theta,$$

$$u^2=V^2+\left(\frac{W}{W+w}.\rho\right)^2-\frac{2W}{W+w}.V\rho\cos\theta.$$

Let the velocities of the two particles referred to any rectangular axes be u_1, v_1 and u_2, v_2, so that

$$u^2=u_1^2+v_1^2,\quad U^2=u_2^2+v_2^2.$$

Then $$\rho^2=(u_2-u_1)^2+(v_2-v_1)^2=U^2+u^2-2(u_1u_2+v_1v_2), \quad\text{.............(i)}$$

$$V^2=\left(\frac{wu_1+Wu_2}{w+W}\right)^2+\left(\frac{wv_1+Wv_2}{w+W}\right)^2=\frac{w^2u^2+W^2U^2+2wW(u_1u_2+v_1v_2)}{(w+W)^2}, \quad\text{....(ii)}$$

and $$\cos\theta=\frac{u_2-u_1}{\rho}\cdot\frac{wu_1+Wu_2}{(W+w)V}+\frac{v_2-v_1}{\rho}\cdot\frac{wv_1+Wv_2}{(W+w)V},$$

i.e. $$(W+w)V\rho\cos\theta=WU^2-wu^2+(w-W)(u_1u_2+v_1v_2). \quad\text{.................(iii)}$$

From (i) and (ii) we get

$$wu^2+WU^2=(w+W)V^2+\frac{wW}{W+w}\cdot\rho^2,$$

and from (i) and (iii),

$$u^2-U^2=-2V\rho\cos\theta+\frac{W-w}{W+w}\cdot\rho^2.$$

Solving these equations, we obtain the required values of u^2 and U^2.

696. [R. 7. b. γ.] *A particle is projected along the inside of a smooth sphere of radius a from its lowest point, so that after leaving the sphere it describes a free path passing through the lowest point. Prove that the velocity of projection is* $\sqrt{\tfrac{7}{2}ag}$.

Suppose the particle leaves the sphere at a point P at an angular distance θ from the highest point. Then, if v be the velocity of projection,

$$\frac{v^2-2ga(1+\cos\theta)}{a}=g\cos\theta,$$

$$\text{i.e. } v^2=ga(2+3\cos\theta).$$

The equation to the path referred to horizontal and vertical axes through P is

$$y=x\tan\theta-\tfrac{1}{2}g\cdot\frac{x^2}{V^2\cos^2\theta},$$

where V is the velocity at P, so that $V^2=ga\cos\theta$.

The co-ordinates of the lowest point are $a\sin\theta$, $-a(1+\cos\theta)$.

Hence $$-(1+\cos\theta)=\sin\theta\tan\theta-\tfrac{1}{2}\cdot\frac{\sin^2\theta}{\cos^3\theta},$$

leading to $$2\cos^3\theta+3\cos^2\theta-1=0,$$

or $$(2\cos\theta-1)(\cos\theta+1)^2=0,$$

whence $\cos\theta=\tfrac{1}{2}$;

$$\therefore\ v^2=ga(2+\tfrac{3}{2})=\tfrac{7}{2}ga.$$

697. [K. 11. a.] *Two circles intersect orthogonally at a point P, and O is any point on any circle which touches the two former circles at Q and Q′. Shew that the angle of intersection of the circles OPQ, OPQ′ is half a right angle.*

Invert the system from P and let q, q', o be the inverses of Q, Q', O. Then the circles become two perpendicular straight lines through q and q', the circle OQQ' becomes a circle touching these lines at q and q' and passing through o, and the circles OPQ, OPQ' become the lines oq and oq'. But if C is the centre of the circle oqq', then $\angle qoq'=\tfrac{1}{2}qCq'$ and qCq' is a right angle. Hence the theorem.

698. [D. b. a.] *Shew that for an angle of* 10° *the value of θ obtained from*

$$\theta=\sin\theta\cdot\sqrt[3]{\sec\theta}$$

is within one second of the truth.

We have

$$\sin\theta\cdot\sqrt[3]{\sec\theta}=\left(\theta-\frac{1}{6}\theta^3+\frac{1}{120}\theta^5-\dots\right)\left(1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4-\dots\right)^{-\frac{1}{3}}$$

$$=\left(\theta-\frac{1}{6}\theta^3+\frac{1}{120}\theta^5-\dots\right)\left(1+\frac{1}{6}\theta^2+\frac{1}{24}\theta^4-\dots\right)$$

$$=\theta+\frac{1}{45}\theta^5+\dots,$$

and it is evident that in the expansion the coefficients of successive powers of θ will continually diminish. Hence the error is less than

$$\frac{1}{45}(\theta^5+\theta^7+\theta^9+\dots \textit{ ad inf.}),\ \textit{i.e.}\ <\frac{1}{45}\,\frac{\theta^5}{1-\theta^2},$$

$$\textit{i.e.}\ <\frac{1}{45}\,\frac{\left(\frac{\pi}{18}\right)^4}{1-\left(\frac{\pi}{18}\right)^2}\times\frac{\pi}{18}\times\frac{180}{\pi}\times 3600 \text{ seconds},$$

and, on calculation, this quantity is slightly less than unity.

699. [K. 6. b; L[1]. 5. a.] *If the normal at α to $\frac{l}{r}=1+e\cos\theta$ meet the curve again in the point β, shew that*

$$\tan\frac{\beta}{2}=-\cot\frac{\alpha}{2}\cdot\frac{1+2e\cos^2\frac{\alpha}{2}+e^2}{1-2e\sin^2\frac{\alpha}{2}+e^2}.$$

The normal at α is

$$\frac{e\sin\alpha}{1+e\cos\alpha}\cdot\frac{l}{r}=e\sin\theta+\sin(\theta-\alpha),$$

and the chord joining α, β is

$$\frac{l}{r}=e\cos\theta+\sec\frac{\alpha-\beta}{2}\cos\left(\theta-\frac{\alpha+\beta}{2}\right).$$

If these coincide we must have

$$\frac{e+\sec\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}}{-\sin\alpha}=\frac{\sec\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}}{e+\cos\alpha}=\frac{1+e\cos\alpha}{\sin\alpha}.$$

From these

$$(e+\cos\alpha)\left[e\left(1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)+1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right]=-\sin\alpha\left(\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}\right),$$

whence

$$\tan\frac{\beta}{2}=-\frac{\sin\alpha\tan\frac{\alpha}{2}+(e+\cos\alpha)(e+1)}{\tan\frac{\alpha}{2}(e-1)(e+\cos\alpha)+\sin\alpha}$$

$$=-\frac{2\sin^2\frac{\alpha}{2}+\left(e+2\cos^2\frac{\alpha}{2}-1\right)(e+1)}{(e-1)\left(e+1-2\sin^2\frac{\alpha}{2}\right)+2\cos^2\frac{\alpha}{2}}\cdot\cot\frac{\alpha}{2}$$

as given.

700. [L^1. 5. a.] *If the normals at P, Q, R to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, meet on the fixed normal at the point whose eccentric angle is α, shew that the sides of the triangle PQR touch the conic*

$$\left(\frac{x}{a}\cos\alpha-\frac{y}{b}\sin\alpha\right)^2+2\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha\right)+1=0.$$

If the normals at α, β, γ are concurrent, then $\Sigma\sin(\beta+\gamma)=0$. Putting $\frac{\beta+\gamma}{2}=\theta$, $\frac{\beta-\gamma}{2}=\phi$, this becomes

$$\sin 2\theta+2\sin(\alpha+\theta)\cos\phi=0.$$

The chord (β, γ) is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=\cos\phi=-\frac{\sin\theta\cos\theta}{\sin(\alpha+\theta)}.$$

Putting $\tan\theta=t$, this becomes

$$t^2.\frac{y}{b}\cos\alpha+t\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha+1\right)+\frac{x}{a}\sin\alpha=0,$$

and the envelope is

$$\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha+1\right)^2=4\frac{xy}{ab}\sin\alpha\cos\alpha.$$

701. [R. 1. a.] *Particles slide from a common vertex down a number of straight tubes each of length l in the same vertical plane. Shew that the locus of the foci of the subsequent parabolic paths consists of portions of two curves whose polar equations may be written*

$$r=l\cos\tfrac{1}{2}\theta \text{ and } r=l\sin\tfrac{1}{2}\theta,$$

the common vertex being the pole.

Let AB, AC be any two of the tubes on opposite sides of the vertical through A. Then the horizontal line through A is the common directrix of all the subsequent parabolic paths. Let BM be the perpendicular from B on this line. Make $\angle ABS=ABM$, and $BS=BM$. Then S is the focus of the path of the particle emerging at B. But $AS=l\cos BAS$. Hence if (r, θ) be the polar co-ordinates of S referred to A as pole, and AM as initial line, we have $$r=l\cos\tfrac{1}{2}\theta, \text{ since } \angle BAS=\tfrac{1}{2}MAS.$$

Similarly for AC the locus will be $r=l\cos\frac{1}{2}(\pi-\theta)$, *i.e.* $r=l\sin\frac{1}{2}\theta$.

702. [K. 2. a.] *Shew that the angle between the pedal lines of any two points on a circle is half the angle subtended by those points at the centre: also that the pedal lines of the extremities of any diameter intersect on the nine-point circle.*

Let LMN be the pedal line of P. Then $\angle PLN=PBN$. Hence the inclinations of the pedal lines of P and Q to BC are $90°-PBA$ and $90°-QBA$, so that the angle between them is PBQ, *i.e.* half the angle subtended by PQ at the centre.

It follows from this that the pedal lines at the extremities of a diameter PQ are at right angles. Let O be the orthocentre, and let OP, OQ meet the pedal lines in Y and Y'. Then Y and Y' are on the nine-point circle, and since they bisect OP and OQ, $\therefore$ $YY'=\frac{1}{2}PQ$, *i.e.* YY' is a diameter of the nine-point circle. But if the pedal lines intersect in V, then YVY' is a right angle; $\therefore$ V is on the nine-point circle.

703. [B. 1. a.] *Prove that the determinant*

$$\begin{vmatrix} a, & b, & c, & b, & a \\ b, & c, & b, & a, & a \\ c, & b, & a, & a, & b \\ b, & a, & a, & b, & c \\ a, & a, & b, & c, & b \end{vmatrix}$$

is equal to $$(2a+2b+c)(a^2+b^2-c^2+bc+ca-3ab)^2.$$

Adding the rows, we may remove the factor $2a+2b+c$. Now subtracting each column from the first, and putting $b-c=\alpha$, etc., the determinant becomes

$$\begin{vmatrix} -\alpha, & 0, & \gamma, & \gamma \\ \alpha, & -\beta, & -\beta, & \alpha \\ \gamma, & \gamma, & 0, & -\alpha \\ 0, & -\gamma, & \beta, & -\gamma \end{vmatrix}$$

Subtracting the third column from the fourth, and remembering that $\alpha+\beta+\gamma=0$, the determinant is

$$\begin{vmatrix} -\alpha, & 0, & \gamma, & 0 \\ \alpha, & -\beta, & -\beta, & -\gamma \\ \gamma, & \gamma, & 0, & -\alpha \\ 0, & -\gamma, & \beta, & \alpha \end{vmatrix}$$

the value of which is easily found to be $(\gamma^2+\alpha\beta)^2$.

704. [K. 8. c.] *A, B, C, D are the vertices of a quadrilateral circumscribing a circle of radius R, and Δ, Δ' are the respective areas of $ABCD$ and the quadrilateral whose vertices are the points of contact of $ABCD$ with the circle. Shew that*

$$\frac{\Delta}{\Delta'}=\frac{OA\,.\,OB\,.\,OC\,.\,OD}{2R^4},$$

where O is the centre of the circle.

Let the angles of the quadrilateral be 2α, 2β, 2γ, 2δ. Then we have

$$\Delta=R^2\,.\,\Sigma\cot\alpha,\quad \Delta'=\tfrac{1}{2}R^2\,.\,\Sigma\sin 2\alpha\,;$$

$$\therefore\quad \frac{\Delta}{\Delta'}=\frac{2\,.\,\Sigma\cot\alpha}{\Sigma\sin 2\alpha}.$$

Now $\Sigma\sin 2\alpha=2\sin(\alpha+\beta)\cos(\alpha-\beta)+2\sin(\gamma+\delta)\cos(\gamma-\delta)$

$=2\sin(\alpha+\beta)[\cos(\alpha-\beta)+\cos(\gamma-\delta)]$, since $\Sigma\alpha=\pi$.

Also $$\Sigma\cot\alpha=\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}+\frac{\sin(\gamma+\delta)}{\sin\gamma\sin\delta}\,;$$

$$\therefore\quad 2\Sigma\cot\alpha=\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta\sin\gamma\sin\delta}\,.\,(2\sin\gamma\sin\delta+2\sin\alpha\sin\beta),$$

and the expression in the bracket is

$$\cos(\gamma-\delta)-\cos(\gamma+\delta)+\cos(\alpha-\beta)-\cos(\alpha+\beta)$$

and $$\cos(\gamma+\delta)=-\cos(\alpha+\beta).$$

Hence $$\frac{\Delta}{\Delta'}=\tfrac{1}{2}\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma\operatorname{cosec}\delta$$

and $$\operatorname{cosec}\alpha=\frac{OA}{R}\text{ etc.}$$

705. [L[1]. 10. b.] *From a point P on a parabola two normals are drawn to the curve. Prove that the bisectors of the angles between these form, with the diameter through P and the normal at P, a harmonic pencil.*

If the normals at m, m' pass through μ, then m, m' are the roots of

$$m^2+\mu m+2=0.$$

The chords through the origin parallel to the normals are

$$(mx+y)(m'x+y)=0, \quad i.e. \ 2x^2-\mu xy+y^2=0,$$

and the bisectors of the angles between these are

$$\frac{x^2-y^2}{1}=\frac{xy}{-\frac{1}{2}\mu},$$

$$i.e. \ \mu(x^2-y^2)+2xy=0, \quad \text{or } (\mu x+y)^2-(1+\mu^2)y^2=0,$$

and these form a harmonic pencil with $\mu x+y=0$ and $y=0$, which are the lines through the origin parallel to the normal at P and the diameter through P respectively.

706. [L[1]. 17. e.] *Prove that the locus of the centres of all conics which have double contact with a given conic, the chords of contact being in a fixed direction, is the diameter of the given conic which is conjugate to the given direction.*

Let the conic be $ax^2+2hxy+by^2=1$, and the fixed direction that of $y=mx$. Then any one of the conics in question is of the form

$$ax^2+2hxy+by^2-1+\lambda(y-mx-c)^2=0,$$

and the centre is given by

$$ax+hy-\lambda m(y-mx-c)=0,$$

$$hx+by+\lambda(y-mx-c)=0.$$

Eliminating λ, we get $\qquad ax+hy+m(hx+by)=0,$

i.e. $y=m'x$, where $\qquad m'=-\dfrac{hm+a}{h+bm},$

or $\qquad a+h(m+m')+bmm'=0.$

But this is the condition that the diameters $y=mx$, $y=m'x$ should be conjugate. Hence the result.

707. [R. 7. b. γ.] *Two equal elastic balls are projected toward each other at the same instant in the same vertical plane, v being the velocity and α the elevation in each case. Shew that after impact they will return to the points of projection if*

$$ga(1+e)=ev^2 \sin 2\alpha,$$

e being the coefficient of elasticity and 2a the distance between the points of projection.

At the impact the horizontal velocity of each is multiplied by e, the vertical velocity remaining unaltered.

Let t be the time to the impact, t' the time of return, so that

$$t=\frac{a}{v\cos\alpha}, \quad t'=\frac{a}{ev\cos\alpha}.$$

Also in time $t+t'$ the vertical distance described by either is zero;

$$\therefore \ v\sin\alpha(t+t')-\tfrac{1}{2}g(t+t')^2=0;$$

$$\therefore \ \frac{2v\sin\alpha}{g}=t+t'=\frac{a}{v\cos\alpha}\left(1+\frac{1}{e}\right),$$

$$i.e. \ ev^2\sin 2\alpha=ga(1+e).$$

708. [R. 7. b. γ.] *A particle hangs attached to one end of an inelastic string of length a, the other end being attached to a fixed point O. The particle is projected so as to move in a vertical circle round O with a velocity due to a height h above O. Prove* (1) *that if $a > \frac{2}{3}h$, the circular motion will cease when the particle has risen to a height $\frac{2}{3}h$ above O;* (2) *that the distance from O of the axis of the subsequent free parabolic path is*

$$a\left(1-\frac{4h^2}{9a^2}\right)^{\frac{3}{2}}.$$

Suppose the circular motion ceases at P, and let S be the focus of the subsequent parabolic path, the tangent at P making an angle θ both with the vertical and with SP.

Since the tension vanishes at P, we have

$$m\,.\,\frac{2g(h-a\sin\theta)}{a}=mg\sin\theta, \quad i.e.\ \sin\theta=\frac{2h}{3a},$$

and the height of P above O is $\quad a\sin\theta=\dfrac{2}{3}h.$

Again, since the velocity at P in the parabola is that due to a fall from the directrix;

$$\therefore\ SP=\text{height of directrix above } P=h-a\sin\theta,$$

and $PN=SP\sin 2\theta$, where PN is perpendicular to the vertical through S.

Hence the distance of O from the axis

$$\begin{aligned}&=a\cos\theta-(h-a\sin\theta)\sin 2\theta\\ &=a\cos\theta-\tfrac{1}{2}a\sin\theta\sin 2\theta=a\cos^3\theta\\ &=a\left(1-\frac{4h^2}{9a^2}\right)^{\frac{3}{2}}.\end{aligned}$$

709. [K. 23; 10. e.] *If a circle cuts the sides of a triangle ABC in the points $X, X' : Y, Y' : Z, Z'$, shew that the triangles formed by the lines $Y'Z, Z'X, X'Y$ and YZ', ZX', XY' are in perspective with ABC, and that the triangles have a common centre of perspective.*

Considering the hexagons $XX'YY'ZZ'$ and $YZ'ZX'XY'$, we see from Pascal's Theorem that each of the triangles in question is in perspective with ABC. If the triangles are $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, then again from Pascal's Theorem, considering the hexagon $Z'ZX'YY'X$, we see that A, α, α' are collinear. Similarly for B, β, β' and C, γ, γ'. Hence the triangles have a common centre of perspective.

710. [J. 1. a.] *A necklace is composed of similar beads of two different sizes, three groups of small ones, each $2n$ in number, being separated by three single larger ones. Shew that the necklace may be restrung in $3n(n+1)$ distinct ways.*

When two large beads are together, the other large one can be in $3n+1$ places (taking account of the cases which are the same on turning over). When two large beads are separated by one small one, to exclude the former case we must add two more small beads one on each side before filling the remaining places. Thus in this case we get $(3n-1)$ necklaces. The next case will give $(3n-2)$, the next $(3n-4)$, the next $(3n-5)$, the next $(3n-7)$, and so on. Hence the total number of necklaces is

$$\begin{aligned}&(3n+1)+(3n-2)+(3n-5)+\ldots+[3n-(3n-1)]\\ &+(3n-1)+(3n-4)+(3n-7)+\ldots+[3n-(\overline{3n-1}+1)],\end{aligned}$$

i.e. $3n^2+3n+1$, and this includes the original one. Hence the number of re-arrangements is $3n(n+1)$.

711. [I. 3. b.] *Shew that if p be a prime, then*

$$(3p)!-6(p!)^3$$

is a multiple of p^6.

We have

$$(3p)!-6(p!)^3=6p^3.(p-1)!\,[(p+1)\dots(2p-1)(2p+1)\dots(3p-1)-\{(p-1)!\}^2]$$
$$=6p^3.(p-1)!\,[(4p^2-1^2)(4p^2-2^2)\dots(4p^2-\overline{p-1}^2)-\{(p-1)!\}^2].$$

Now if p is prime and

$$(x+1)(x+2)\dots(x+p-1)=x^{p-1}+A_1x^{p-2}+\dots+A_{p-2}x+(p-1)!$$

then, by Lagrange's Theorem, the coefficients A are all multiples of p.

Also, since $(p-1)$ is even, we have

$$(x-1)(x-2)\dots(x-\overline{p-1})=x^{p-1}-A_1x^{p-2}\dots-A_{p-2}x+(p-1)!.$$

Hence multiplying

$$(x^2-1^2)(x^2-2^2)\dots(x^2-\overline{p-1}^2)-\{(p-1)!\}^2$$
$$=x^{2p-2}+\dots+\{2A_{p-3}(p-1)!-A_{p-2}{}^2\}x^2,$$

and the coefficient of x^2 on the right is a multiple of p. Hence putting $x=2p$, we see that the expression in the square bracket above is a multiple of p^3. Hence the original expression is a multiple of p^6.

712. [K. 20. d.] *If*

$$\cos(x+\theta)\cos(y+\theta)\cos(z+\theta)+\sin(x'-\theta)\sin(y'-\theta)\sin(z'-\theta)=0,$$

and

$$x+y+z+x'+y'+z'=\frac{\pi}{2},$$

then will

$$\tan\theta=\frac{\cos x\cos y\cos z+\sin x'\sin y'\sin z'}{\cos x'\cos y'\cos z'+\sin x\sin y\sin z}.$$

Putting $\tan\theta=t$, the given equation is

$$\Pi(\cos x-t\sin x)+\Pi(\sin x'-t\cos x')=0.$$

The coefficient of t is

$$-(\Sigma\sin x\cos y\cos z+\Sigma\cos x'\sin y'\sin z')$$
$$=-\sin(x+y+z)-\sin x\sin y\sin z-\cos x'\cos y'\cos z'+\cos(x'+y'+z')$$
$$=-(\sin x\sin y\sin z+\cos x'\cos y'\cos z').$$

Similarly the coefficient of t^2 is $\cos x\cos y\cos z+\sin x'\sin y'\sin z'$.

Hence the equation is

$$(\cos x\cos y\cos z+\sin x'\sin y'\sin z')(1+t^2)$$
$$-(\sin x\sin y\sin z+\cos x'\cos y'\cos z')(t+t^3)=0.$$

Dividing by $1+t^2$, this gives the required value of t.

713. [L^1. 4. c.] *Shew that the locus of the point such that the tangents from it to the ellipse of semi-major axis a make an angle 2α with each other is given by the equation*

$$a_1{}^2\cos^2\alpha+a_2{}^2\sin^2\alpha=a^2,$$

where a_1 *and* a_2 *are the primary semi-axes of the confocals through the point.*

If $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ is a confocal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ through (X, Y), we must have

$$\lambda^2-\lambda(X^2+Y^2-a^2-b^2)-(b^2X^2+a^2Y^2-a^2b^2)=0.$$

Hence $$\tan 2\alpha = \frac{2\sqrt{b^2X^2 + a^2Y^2 - a^2b^2}}{X^2 + Y^2 - a^2 - b^2}$$

$$= \frac{2\sqrt{-\lambda_1\lambda_2}}{\lambda_1 + \lambda_2}.$$

Now $\lambda_1 = a_1^2 - a^2$, $\lambda_2 = a_2^2 - a^2$, so that $\tan 2\alpha = \dfrac{2\sqrt{(a_1^2 - a^2)(a^2 - a_2^2)}}{a_1^2 - a^2 - (a^2 - a_2^2)}$.

But $\tan 2\alpha = \dfrac{2pq}{p^2 - q^2}$ gives $\tan\alpha = \dfrac{q}{p}$;

$$\therefore\ \tan\alpha = \sqrt{\frac{a^2 - a_2^2}{a_1^2 - a^2}};\quad \therefore\ \frac{\sin^2\alpha}{a^2 - a_2^2} = \frac{\cos^2\alpha}{a_1^2 - a^2} = \frac{1}{a_1^2 - a_2^2},$$

whence $$a_1^2\sin^2\alpha + a_2^2\cos^2\alpha = a^2.$$

714. [R. 7. b. γ.] *A shot is to be fired so as to enter horizontally with a given velocity a wooden partition perpendicular to the plane of its path, and, on emerging, to hit an object at the same level as the gun on the other side. The object is at a distance a and the partition at a distance d from the point of projection and the thickness of the partition, supposed small, is b. It is found, however, that the shot falls short of the object by a small distance c. Shew that, if the object is to be hit, the thickness of the partition must be diminished by*

$$\frac{2(a-d)bc}{a(2d-a)},\ \textit{approximately},$$

the resistance of the wood being supposed uniform.

Let f be the negative acceleration due to the resistance, u the velocity with which the shot enters the partition at height h.

Then $$h = \tfrac{1}{2}g\cdot\frac{d^2}{u^2} = \tfrac{1}{2}g\cdot\frac{x^2}{u^2 - 2bf},$$

where $$x = a - c - b - d,$$

$$\textit{i.e.}\ \frac{2bf}{u^2} = 1 - \frac{x^2}{d^2}.$$

Similarly in the second case, when the thickness is diminished by t, we have

$$\frac{2(b-t)f}{u^2} = 1 - \frac{(x+c)^2}{d^2}.$$

Hence $$\frac{b}{b-t} = \frac{d^2 - x^2}{d^2 - (x+c)^2};$$

$$\therefore\ \frac{t}{b} = \frac{(x+c)^2 - x^2}{d^2 - x^2} = \frac{2cx}{d^2 - x^2}$$

$$= \frac{2c(a-d)}{d^2 - (a-d)^2},$$

neglecting squares of small quantities.

715. [K. 10. e.] *Through one point of intersection of two circles a line is drawn. The points in which it meets the circles are joined to their other point of intersection. Prove that the orthocentre of the triangle so formed lies on a fixed circle.*

Let A and B be the points of intersection, ACD the line. Then since A and B are fixed, the angles ACB and ADB are given, *i.e.* the triangle BCD

is of constant shape. Let BN be the perpendicular on CD, O the orthocentre. Then for a triangle of given shape the ratio $BO : BN$ is evidently constant.

But the locus of N is a circle, viz. the circle on AB as diameter. Hence the locus of O is also a circle.

716. [K. 2. b.] *The internal common tangents (other than the sides) to the inscribed and each of the escribed circles of a triangle ABC are drawn. Shew that the vertices of the triangle formed by them lie in the perpendiculars from the centre of the inscribed circle on the sides of ABC: that the area of the triangle is $r^2 \tan A \tan B \tan C$, and that the radius of its circumcircle is*

$$\tfrac{1}{4} r \sec A \sec B \sec C.$$

Let AII_1 meet BC in D. Then if $A'B'C'$ be the triangle formed by the remaining common tangents, $B'C'$ passes through D and BC, $B'C'$ make equal angles with II_1. Hence evidently $A'B'$ and $A'C'$ make equal angles with BC, so that the triangle formed by these three lines is isosceles, and has the same inscribed circle as ABC.

$$\therefore \ IA' \text{ is perpendicular to } BC. \quad \text{Also } \hat{A}' = \pi - 2A\,;$$

$$\therefore \ a' = B'C' = r\left(\cot\frac{B'}{2} + \cot\frac{C'}{2}\right)$$

$$= r(\tan B + \tan C) = r \cdot \frac{\sin A}{\cos B \cos C}.$$

Hence the area is

$$\tfrac{1}{2} b'c' \sin A' = \tfrac{1}{2} r^2 \cdot \frac{\sin B}{\cos C \cos A} \cdot \frac{\sin C}{\cos A \cos B} \cdot \sin 2A$$

$$= r^2 \tan A \tan B \tan C,$$

and the radius of the circumcircle is

$$\frac{a'}{2 \sin A'} = \tfrac{1}{2} r \cdot \frac{\sin A}{\cos B \cos C} \cdot \frac{1}{\sin 2A} = \tfrac{1}{4} r \sec A \sec B \sec C.$$

717. [D. b. a.] *Determine m and n so that the formula*

$$\frac{\sin x}{x} = \cos^m \tfrac{1}{2} x \,.\, \cos^n x$$

may be approximately true when x is so small that higher powers than the fourth may be neglected.

On the given hypothesis, we have

$$1 - \frac{x^2}{6} + \frac{x^4}{120} = \left(1 - \frac{x^2}{8} + \frac{x^4}{384}\right)^m \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)^n.$$

Expanding the right-hand members, and taking the product as far as x^4, we obtain on reduction

$$1 - (\tfrac{1}{2} n + \tfrac{1}{8} m) x^2 - \left[\frac{3mn(m+4n) - 58mn}{768}\right] x^4.$$

We must therefore have

$$\tfrac{1}{2} n + \tfrac{1}{8} m = \tfrac{1}{6}, \quad i.e. \ 4n + m = \tfrac{4}{3}. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Thence, equating cœfficients of x^4, we find

$$mn = \tfrac{16}{135}. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

The simplest values giving an approximate solution of (i) and (ii) are

$$m = \tfrac{8}{9}, \ n = \tfrac{1}{9}.$$

718. [L[1]. 17. e.] *If two ellipses have perpendicular latera-recta $2l$, $2l'$ and a common focus S, and touch at a distance c from S, prove that their eccentricities are*

$$\sqrt{\left(1-\frac{l}{l'}\right)\left(1-\frac{l}{c}\right)} \text{ and } \sqrt{\left(1-\frac{l'}{l}\right)\left(1-\frac{l'}{c}\right)}.$$

Let the conics be

$$\frac{l}{r}=1+e\cos\theta,\quad \frac{l'}{r}=1+e'\sin\theta,$$

and let α be the vectorial angle corresponding to $r=c$.

The tangents at α are

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha),\quad \frac{l'}{r}=e'\sin\theta+\cos(\theta-\alpha).$$

If these coincide

$$\frac{e+\cos\alpha}{\cos\alpha}=\frac{\sin\alpha}{e'+\sin\alpha}=\frac{l}{l'};$$

$$\therefore\ e=\left(\frac{l}{l'}-1\right)\cos\alpha,\quad e'=\left(\frac{l'}{l}-1\right)\sin\alpha.$$

But $$e\cos\alpha=\frac{l}{c}-1,\quad e'\sin\alpha=\frac{l'}{c}-1;$$

$$\therefore\ e^2=\left(\frac{l}{l'}-1\right)\left(\frac{l}{c}-1\right),\quad e'^2=\left(\frac{l'}{l}-1\right)\left(\frac{l'}{c}-1\right).$$

719. [L[1]. 11. b.] *Shew that the equation of the circle of curvature at any point $(x'y')$ of a rectangular hyperbola is*

$$x^2+y^2-\frac{x}{x'}(3x'^2+y'^2)-\frac{y}{y'}(3y'^2+x'^2)+3(x'^2+y'^2)=0,$$

and hence shew that the centre of curvature divides the normal chord externally in the ratio 1 : 3, *the asymptotes being the axes of reference.*

The tangent at (x', y') is $\frac{x}{x'}+\frac{y}{y'}=2$, and $\therefore$ the circle of curvature is of the form

$$xy-x'y'+\lambda\left(\frac{x}{x'}+\frac{y}{y'}-2\right)(px+qy-1)=0,$$

where $px'+qy'=1$. The conditions for a circle are

$$\frac{p}{x'}=\frac{q}{y'},\quad 1+\lambda\left(\frac{q}{x'}+\frac{p}{y'}\right)=0,$$

whence $$\frac{p}{x'}=\frac{q}{y'}=\frac{1}{x'^2+y'^2},\quad \lambda=-x'y'.$$

Substituting and reducing we obtain the given equation.

Further, the normal is $xx'-yy'=x'^2-y'^2$, which meets the hyperbola again at $\left(-\frac{y'^2}{x'}, -\frac{x'^2}{y'}\right)$, and the centre of curvature being $\frac{1}{2}\left(3x'+\frac{y'^2}{x'}\right)$, $\frac{1}{2}\left(3y'+\frac{x'^2}{y'}\right)$, the second result is evident.

720. [L^1. 17. a.] *Prove that the four common tangents to the conics*

$$l\alpha^2+m\beta^2+n\gamma^2=0, \quad l'\alpha^2=m'\beta^2+n'\gamma^2=0,$$

are the lines

$$\sqrt{ll'(mn'-m'n)}\,.\,\alpha\pm\sqrt{mm'(nl'-n'l)}\,.\,\beta\pm\sqrt{nn'(lm'-l'm)}\,.\,\gamma=0,$$

and that their points of contact lie on the conic

$$ll'(mn'+m'n)\alpha^2+mm'(nl'+n'l)\beta^2+nn'(lm'+l'm)\gamma^2=0.$$

The conditions that $\lambda\alpha+\mu\beta+\nu\gamma=0$ should be a tangent to both conics are $\Sigma\frac{\lambda^2}{l}=0$, $\Sigma\frac{\lambda^2}{l'}=0$, whence solving

$$\frac{\lambda^2}{\frac{1}{mn'}-\frac{1}{m'n}}=\ldots=\ldots,$$

$$i.e. \quad \frac{\lambda^2}{ll'(mn'-m'n)}=\ldots=\ldots.$$

The points of contact are $\left(\frac{\lambda}{l},\ldots\right)$ and $\left(\frac{\lambda}{l'},\ldots\right)$, and from the identities

$$\Sigma l^2(m^2n'^2-m'^2n^2)=0, \quad \Sigma l'^2(m^2n'^2-m'^2n^2)=0,$$

it is evident that these lie on

$$\Sigma ll'(mn'+m'n)\alpha^2=0.$$

721. [R. 1. e.] *Five rods in the same plane are smoothly jointed together in the form of two triangles ABC, DBC on the same base BC, and on the same side of it, AD being parallel to BC. The middle points of AC and BD are joined by a string at tension T. Shew that the stress in BC is*

$$\frac{AD+BC}{2BC}\,.\,T.$$

The reaction at A must be in the direction AB. Hence if the direction of the string meets AB, CD in H, K, then considering the equilibrium of the rod AC, the reaction at C must be in the direction CH. Hence, drawing BG $\parallel$ to CH meeting DC produced in G, then BCG is the triangle of forces for the joint C. Hence if R is the stress in BC, S the force along CH, we have $\frac{R}{S}=\frac{BC}{BG}$. Also BCH is the force-triangle for the rod AC; $\therefore$ $\frac{T}{S}=\frac{BC}{CH}$.

Now let BA, CD meet in O. Then

$$\frac{R}{T}=\frac{CH}{BG}=\frac{OH}{OB}=\frac{HK}{BC}=\frac{1}{2}\cdot\frac{AD+BC}{BC},$$

since H, K are the middle points of AB, CD.

722. [R. 9. b.] *A particle of mass m is attached by two inelastic strings to particles of masses m′ and m″. The particles are placed at rest on a smooth horizontal table, so that the two strings lie in perpendicular straight lines. A blow is given to the particle m in the direction of the bisector of the angle between the strings. Shew that the initial velocities of m′ and m″ are in the ratio*

$$m+m'' : m+m'.$$

Let T_1, T_2 be the impulsive tensions, u and v the velocities of m after impact $\parallel$ and perpendicular to its original direction. Then the initial velocity of m' is

$$u \cos 45^\circ - v \sin 45^\circ = \frac{u-v}{\sqrt{2}},$$

and that of m'' is $\frac{u+v}{\sqrt{2}}$. We $\therefore$ have

$$(T_1 - T_2) \cos 45^\circ = mv, \quad T_1 = m' . \frac{u-v}{\sqrt{2}}, \quad T_2 = m'' . \frac{u+v}{\sqrt{2}},$$

whence

$$\frac{u}{v} = \frac{m' + m'' + 2m}{m' - m''},$$

$$i.e. \quad \frac{u-v}{u+v} = \frac{m''+m}{m'+m}.$$

723. [A. 1. b.] *If a be positive and not equal to unity, then*

$$\frac{1 + a^2 + a^4 + \dots + a^{2n}}{a + a^3 + a^5 + \dots + a^{2n-1}} > \frac{n+1}{n}.$$

We have to shew that

$$n(1 + a^2 + \dots + a^{2n}) - (n+1)(a + a^3 + a^5 + \dots + a^{2n-1})$$

is positive. Calling this expression $f(n)$, we have

$$f(n) - af(n-1) = n(1 + 2a^2 + 2a^4 + \dots + 2a^{2n-2} + a^{2n}) - 2n(a + a^3 + \dots + a^{2n-1})$$
$$= n(1-a)(1 - a + a^2 - a^3 - \dots - a^{2n-1})$$
$$= n(1-a)^2(1 + a^2 + a^4 + \dots + a^{2n-2}),$$

an essentially positive quantity. Hence if $f(n-1)$ is positive, then $f(n)$ must be positive. But $f(1) = (1-a)^2$. Hence the result follows by induction.

724. [J. 2. c.] *From three bags each containing tickets numbered 1 to p, three tickets are drawn, one from each. Shew that the chance that the sum of the numbers on them is $2p$ is*

$$\frac{(p-1)(p+4)}{2p^3}.$$

The total number of possibilities is p^3, and the number of cases in which the sum is $2p$ is the coefficient of x^{2p} in $(x^1 + x^2 + \dots + x^p)^3$, *i.e.* the coefficient of x^{2p-3} in $(1-x^p)^3(1-x)^{-3}$, which is

$$-3 . \frac{(p-1)(p-2)}{2!} + \frac{(2p-2)(2p-1)}{2!} = \frac{(p-1)(p+4)}{2}.$$

Hence the required chance is

$$\frac{(p-1)(p+4)}{2p^3}.$$

725. [K. 20. d.] *If $\alpha + \beta + \gamma = \pi$, and*

$$\cos(x+\alpha)\cos(x+\beta)\cos(x+\gamma) + \cos^3 x = 0,$$

then

$$\tan x = \cot\alpha + \cot\beta + \cot\gamma,$$
$$\tan^2 x = 2 + \cot^2\alpha + \cot^2\beta + \cot^2\gamma.$$

Putting $\tan x = t$, we have

$$(\cos\alpha - t\sin\alpha)(\cos\beta - t\sin\beta)(\cos\gamma - t\sin\gamma) + 1 = 0,$$

i.e. $\cos\alpha\cos\beta\cos\gamma - t.\Sigma\cos\beta\cos\gamma\sin\alpha$
$$+t^2.\Sigma\cos\alpha\sin\beta\sin\gamma - t^3\sin\alpha\sin\beta\sin\gamma+1=0.$$

Now $\sin(\alpha+\beta+\gamma)=0$; $\therefore\ \Sigma\cos\beta\cos\gamma\sin\alpha-\sin\alpha\sin\beta\sin\gamma=0$,
and $\cos(\alpha+\beta+\gamma)=-1$; $\therefore\ \cos\alpha\cos\beta\cos\gamma-\Sigma\cos\alpha\sin\beta\sin\gamma+1=0$.

Hence the above equation becomes
$$(1+t^2).\Sigma\cos\alpha\sin\beta\sin\gamma-(t+t^3)\sin\alpha\sin\beta\sin\gamma=0;$$
$$\therefore\ t=\frac{\Sigma\cos\alpha\sin\beta\sin\gamma}{\sin\alpha\sin\beta\sin\gamma}=\Sigma\cot\alpha.$$

Hence also $\qquad t^2=\Sigma\cot^2\alpha+2\Sigma\cot\beta\cot\gamma,$
and $\Sigma\cot\beta\cot\gamma=1$, since $\alpha+\beta+\gamma=\pi$.

726. [K. 6. b.] *Prove that the locus of the point*
$$x=\frac{at^2+bt+c}{At^2+Bt+C},\qquad y=\frac{a't^2+b't+c'}{At^2+Bt+C},$$
where t is a variable parameter, will be a straight line if
$$A(bc'-b'c)+B(ca'-c'a)+C(ab'-a'b)=0,$$
but that it will be necessary to include imaginary values of t if the point is to trace out the whole line.

For example, find the Cartesian equation of the line
$$x=a.\frac{t^2+2t-5}{2t^2-6t+5},\qquad y=a.\frac{3t^2-8t+6}{2t^2-6t+5},$$
and shew that real values of t correspond to a length $\sqrt{26}.a$ of the line.

Suppose the Cartesian equation to the line is
$$lx+my+n=0.$$
Then we must have
$$l(at^2+bt+c)+m(a't^2+b't+c')+n(At^2+Bt+C)=0$$
for all values of t.

Hence
$$la+ma'+nA=0,$$
$$lb+mb'+nB=0,$$
$$lc+mc'+nC=0.$$

Eliminating l, m, n from these, we get the required condition.

In the given case $\frac{l}{1}=\frac{m}{-5}=\frac{n}{7a}$, and the condition is satisfied, the Cartesian equation being $x-5y+7a=0$.

We also have
$$(2x-a)t^2-(6x+2a)t+5(x+a)=0,$$
and if t is real, this implies that
$$(3x+a)^2-5(2x-a)(x+a) \text{ is positive,}$$
$$\text{i.e. } x^2-ax-6a^2 \text{ is negative,}$$
so that x lies between $3a$ and $-2a$, the corresponding values of y being $2a$ and a. Hence real values of t correspond to a length
$$\sqrt{(3a+2a)^2+(2a-a)^2}=\sqrt{26}.a.$$

727. [L¹. 16. a.] *Prove that the general equation of a conic inscribed in the quadrilateral $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$ may be written*

$$(\mu-\nu)^2(\beta\gamma+\alpha\delta)+(\nu-\lambda)^2(\gamma\alpha+\beta\delta)+(\lambda-\mu)^2(\alpha\beta+\gamma\delta)=0.$$

The equation to any conic inscribed in the triangle of reference is

$$\Sigma l^2\alpha^2-2\Sigma mn\beta\gamma=0, \qquad \text{(i)}$$

and if $P\delta=L\alpha+M\beta+N\gamma$, we have the condition $\Sigma\frac{l}{L}=0$.

Now $\alpha^2=\dfrac{P\alpha\delta-M\alpha\beta-N\alpha\gamma}{L}$, etc., and substituting these in (i), the coefficient of $\beta\gamma$ is

$$-\frac{m^2N}{M}-\frac{n^2M}{N}-2mn=-MN\left(\frac{m}{M}+\frac{n}{N}\right)^2=-MN\frac{l^2}{L^2},$$

and the equation to the conic is

$$P\Sigma\frac{l^2}{L}\cdot\alpha\delta-\Sigma MN\frac{l^2}{L^2}\cdot\beta\gamma=0.$$

Now we may replace α, β, γ by any multiples of these quantities, and the multiplier P is also arbitrary. Hence replacing α, β, γ by $\frac{\alpha}{L}$, $\frac{\beta}{M}$, $\frac{\gamma}{N}$ and taking $P=-1$, the equation becomes

$$\Sigma\frac{l^2}{L^2}(\alpha\delta+\beta\gamma)=0,$$

and since $\Sigma\frac{l}{L}=0$, we may take $\frac{l}{L}=\mu-\nu$, etc.

728. [R. 9. b.] *The centres of two inelastic balls in contact are B and C. A third ball, centre A, strikes the two simultaneously and in such a way that the direction of motion of A is unchanged by the impact, and there is no impact between B and C. Shew that if β, γ be the angles which the direction of motion of A makes with AB, AC respectively, and if M_2, M_3 be the masses of B and C, then*

$$M_2\sin 2\beta=M_3\sin 2\gamma,$$

and that if v be the final, u the initial velocity of A, and M_1 its mass, then

$$(M_1+M_2\cos^2\beta+M_3\cos^2\gamma)v=M_1u.$$

Let I, I' be the impulses between the striking sphere and the spheres at rest. Then since the direction of motion is unaltered, we must have

$$M_1(u-v)=I\cos\beta+I'\cos\gamma, \qquad \text{(i)}$$

$$I\sin\beta-I'\sin\gamma=0. \qquad \text{(ii)}$$

Also the initial velocities of the other spheres are $v\cos\beta$, $v\cos\gamma$;

$$\therefore\ I=M_2v\cos\beta,\ I'=M_3v\cos\gamma.$$

Substituting these values in (i) and (ii) the results follow.

729. [R. 9. b.] *Two balls A and B are moving in the same straight line with equal velocities and B impinges directly on a wall. Shew that there will be at least two impacts between A and B if the ratio of their masses is greater than $2e:1+e^2$, where e is the coefficient of elasticity between the balls and the*

wall. Shew also that if the masses are equal, there will be at least three impacts between the balls if $e<2-\sqrt{3}$.

Let u be the initial velocity of each ball. Then the velocity of B after the first impact with the wall is $-eu$.

Hence, if u_1, v_1 are the velocities of A, B after the first impact, m and m' their masses, we have

$$mu_1+m'v_1=mu+m'(-eu),$$

$$u_1-v_1=-e(1+e)u,$$

whence $$u_1=\frac{m-(2e+e^2)m'}{m+m'}\cdot u,\quad v_1=\frac{(1+e+e^2)m-em'}{m+m'}\cdot u.$$

There will be a second impact between the balls if $ev_1>-u_1$,

$$\text{i.e.}\quad e(1+e+e^2)m-e^2m'>-m+(2e+e^2)m',$$

$$\text{i.e.}\quad (1+e+e^2+e^3)m>(2e+2e^2)m',$$

$$\text{i.e.}\quad \frac{m}{m'}>\frac{2e}{1+e^2}.$$

If the masses are equal, and if their velocities after the second impact be u_2 and v_2, we have

$$u_2+v_2=u_1-ev_1=\frac{1-3e-e^2-e^3}{2}\cdot u,$$

$$u_2-v_2=-e(u_1+ev_1)=-\frac{e-e^2-e^3+e^4}{2}\cdot u,$$

whence $$u_2=\frac{1-4e-e^4}{4}\cdot u,\quad v_2=\frac{1-2e-2e^2-2e^3+e^4}{4}\cdot u,$$

and there will be a third impact between the balls if $ev_2>-u_2$,

$$\text{i.e.}\quad e(1-2e-2e^2-2e^3+e^4)>-1+4e+e^4,$$

$$\text{i.e.}\quad 1-3e-2e^2-2e^3-3e^4+e^5>0,$$

$$\text{i.e.}\quad (1+e+e^2+e^3)(1-4e+e^2)>0,$$

$$\text{i.e.}\quad [e-(2+\sqrt{3})][e-(2-\sqrt{3})]>0,$$

$$\text{i.e.}\quad e<2-\sqrt{3}.$$

730. [L[1]. 1. a.] *If two chords are drawn from any point on a conic equally inclined to the normal at that point, the tangents at their further extremities will intersect on the normal.*

Let PQ, PQ' be equally inclined to the normal at P. Let the normal and tangent at P meet QQ' in U', U respectively. Then since PU, PU' are the bisectors of the angle QPQ'; $\therefore$ $P(UU', QQ')$ is harmonic; $\therefore$ the polar of U passes through U', *i.e.* PU' is the polar of U. But if T is the pole of QQ', then PT is the polar of U. Hence PT, PU' coincide, *i.e.* T lies on the normal at P.

731. [J. 1. a.] *If* $2n$ *things be divided into* n *pairs, shew that the number of ways in which they may be re-divided so that no couple of things which were together in the first division shall be together in the second is*

$$\phi(n)-n\phi(n-1)+\frac{n(n-1)}{2!}\cdot\phi(n-2)\ldots+(-1)^{n-1}n\phi(1)+(-1)^n,$$

where $$\phi(n)=1\cdot3\cdot5\ldots(2n-1).$$

The number of ways in which $2n$ things can be divided into n pairs is

$$\frac{(2n)!}{2^n \, . \, n!} = \phi(n).$$

Now, take one arrangement as fixed, and consider the number of possible re-arrangements. If we keep the first pair fixed, we can re-divide the other pairs in $\phi(n-1)$ ways.

Hence the number of re-arrangements in which the first pair is not the same is $\phi(n) - \phi(n-1)$.

Among these there are $\phi(n-1) - \phi(n-2)$ in which the second pair is the same. Hence the number in which neither the first nor the second pair is the same is

$$\phi(n) - 2\phi(n-1) + \phi(n-2).$$

Among these there are $\phi(n-1) - 2\phi(n-2) + \phi(n-3)$ in which the third pair is the same. Hence the number in which neither the first, second nor third pairs are the same is

$$\phi(n) - 3\phi(n-1) + 3\phi(n-2) - \phi(n-3),$$

and so on, the coefficients following the law of the Binomial Theorem, the given expression being the number of cases in which none of the n pairs is the same as before.

732. [A. 1. b.] *If $ax + by + cz = x + y + z$, and the quantities involved are all positive, shew that*

$$a^{ax} b^{by} c^{cz} > 1.$$

Let d be such that axd, byd, czd are integers. Then writing X for xd, etc. we have

$$aX + bY + cZ = X + Y + Z.$$

Now choose aX quantities equal to $\frac{1}{a}$, bX equal to $\frac{1}{b}$, and so on. Then since the A.M. of these quantities is greater than their G.M. we have

$$\frac{X+Y+Z}{aX+bY+cZ} > \left\{\left(\frac{1}{a}\right)^{aX}\left(\frac{1}{b}\right)^{bY}\left(\frac{1}{c}\right)^{cZ}\right\}^{\frac{1}{aX+bY+cZ}},$$

$$i.e. \quad 1 > \left(\frac{1}{a}\right)^{aX}\left(\frac{1}{b}\right)^{bY}\left(\frac{1}{c}\right)^{cZ},$$

$$i.e. \quad 1 > \left(\frac{1}{a}\right)^{ax}\left(\frac{1}{b}\right)^{by}\left(\frac{1}{c}\right)^{cz},$$

$$i.e. \quad a^{ax} b^{by} c^{cz} > 1.$$

733. [L[1]. 10. d.] *Shew that the focus of the parabola*

$$(x\sqrt{a} + y\sqrt{b})^2(a+b) + 2(gx+fy)(a+b) + g^2 + f^2 = 0$$

is at the point

$$-g(a+b)^{-1}, \qquad -f(a+b)^{-1}.$$

The directrix being perpendicular to the axis, let its equation be $x\sqrt{b} - y\sqrt{a} + k = 0$, the focus being (X, Y).

Then the equation to the parabola is

$$(x-X)^2 + (y-Y)^2 = \frac{(x\sqrt{b} - y\sqrt{a} + k)^2}{b+a},$$

$$i.e. \quad (x\sqrt{a} + y\sqrt{b})^2 - 2x[(a+b)X + k\sqrt{b}] - 2y[(a+b)Y - k\sqrt{a}]$$
$$+ (X^2 + Y^2)(a+b) - k^2 = 0.$$

Comparing this with the given equation, we have

$$(a+b)X+k\sqrt{b}=-g, \qquad (X^2+Y^2)(a+b)-k^2=\frac{g^2+f^2}{a+b}.$$
$$(a+b)Y-k\sqrt{a}=-f,$$

Substituting for X, Y in the last equation we find $k=0$, giving the required values of X and Y.

734. [L¹. 4. c.] *Prove that the radius of the circumcircle of the triangle formed by tangents from (h, k) to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and their chord is*

$$\frac{1}{2}\cdot\frac{\sqrt{b^4h^2+a^4k^2}}{b^2h^2+a^2k^2}\cdot\rho\rho'$$

where ρ, ρ' are the distances from (h, k) to the foci.

Let α be the angle between the tangents, l the length of the chord of contact, then the radius of the circle is $l/2\sin\alpha$.

Now $$\tan\alpha=\frac{2\sqrt{b^2h^2+a^2k^2-a^2b^2}}{h^2+k^2-a^2-b^2}.$$

Also $$4(b^2h^2+a^2k^2-a^2b^2)+(h^2+k^2-a^2-b^2)^2$$
$$\equiv(h^2+k^2+a^2-b^2)^2-4h^2(a^2-b^2)$$
$$=\rho^2\rho'^2,$$

since $$\rho^2=(h-ae)^2+k^2, \quad \rho'^2=(h+ae)^2+k^2.$$

Hence $$\sin\alpha=\frac{2\sqrt{b^2h^2+a^2k^2-a^2b^2}}{\rho\rho'}. \quad \text{..........................(i)}$$

Also the x-coordinates of the intersections of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{xh}{a^2}+\frac{yk}{b^2}=1$ are the roots of

$$(b^2h^2+a^2k^2)x^2-2a^2b^2hx+a^4(b^2-k^2)=0,$$

whence $$(x_1-x_2)^2=\frac{4a^4k^2(b^2h^2+a^2k^2-a^2b^2)}{(b^2h^2+a^2k^2)^2}.$$

Writing down the corresponding value of $(y_1-y_2)^2$ and adding we have

$$l^2=\frac{4(b^4h^2+a^4k^2)(b^2h^2+a^2k^2-a^2b^2)}{(b^2h^2+a^2k^2)^2}. \quad \text{........................(ii)}$$

From (i) and (ii) we obtain the given value of $\frac{l}{2\sin\alpha}$.

735. [L¹. 17. e.] *Two conics inscribed in the triangle ABC touch the sides in DEF, $D'E'F'$. Shew that if EF, $E'F'$ meet in P; FD, $F'D'$ in Q, and DE, $D'E'$ in R, then AP, BQ, CR are concurrent, and if $(\alpha', \beta', \gamma')$ be the point of concurrence, then the fourth common tangent to the two conics is*

$$\frac{\alpha}{\alpha'}+\frac{\beta}{\beta'}+\frac{\gamma}{\gamma'}=0.$$

Let the conics be $\sqrt{l\alpha}+\ldots+\ldots=0$, $\sqrt{l'\alpha}+\ldots+\ldots=0$.

Then the equations to EF, $E'F'$ are

$$-l\alpha+m\beta+n\gamma=0, \quad -l'\alpha+m'\beta+n'\gamma=0,$$

and since AP joins the intersection of these to the point of reference A, its equation is $$(lm'-l'm)\beta-(nl'-n'l)\gamma=0.$$

Writing down the corresponding equations of BQ, CR from symmetry, we see that the three lines meet in the point

$$\frac{\alpha}{mn'-m'n}=\ldots=\ldots.$$

If the fourth common tangent is $L\alpha+M\beta+N\gamma=0$, then

$$\Sigma\frac{l}{L}=0 \text{ and } \Sigma\frac{l'}{L}=0,$$

whence
$$\frac{1/L}{mn'-m'n}=\ldots=\ldots,$$

$$i.e. \quad \frac{L}{1/\alpha'}=\frac{M}{1/\beta'}=\frac{N}{1/\gamma'}.$$

736. [R. 4. a.] *A circular ring of radius a and weight W is suspended horizontally by three vertical threads, each of length l, attached to points on its circumference. Prove that, in order to hold the ring horizontal, and twisted through an angle θ, a horizontal couple must be applied equal to*

$$W.\frac{a^2 \sin\theta}{\left(l^2-4a^2\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}}.$$

Suppose that, in the twisted position, each string makes an angle ϕ with the vertical. Let A, A' be the upper and lower ends of one of the strings, O the centre of the circle through the upper ends of the strings, O' the centre of the ring, V and L the middle points of AA' and OO'. Draw AK vertical to meet the ring in K, and let N be the middle point of $A'K$.

Then
$$l\sin\phi=A'K=2a\sin\frac{\theta}{2}.$$

Now evidently VL is the shortest distance between AA' and OO' and

$$VL=O'N=a\cos\frac{\theta}{2}.$$

Resolving vertically $3T\cos\phi=W$, and the moment of the required couple is

$$3T\sin\phi\,.\,VL=W\tan\phi\,.\,a\cos\frac{\theta}{2}$$

$$=W.\frac{2a\sin\frac{\theta}{2}}{\sqrt{l^2-4a^2\sin^2\frac{\theta}{2}}}.\,a\cos\frac{\theta}{2}$$

$$=W.\frac{a^2\sin\theta}{\sqrt{l^2-4a^2\sin^2\frac{\theta}{2}}}.$$

737. [R. 7. f.] *A smooth tube in the form of a parabola is placed with its axis vertical and vertex downwards and a heavy particle is dropped from the highest point A. If the length of the tube be varied by cutting off portions from the end B, shew that the envelope of the different parabolic paths described after emerging from B is another parabola.*

We suppose the end B to be above the vertex. The directrix of the free path is evidently the horizontal through A, since the velocity at B is that

due to the vertical distance between A and B. Let S' be the focus of the free path, and let BS' meet the path again in P. Draw PN vertical to meet the directrix in N, and produce it to N' so that NN' = the distance between A and the directrix of the tube. Then the horizontal through N' is a fixed line. Also since the parabolas have a common tangent at B; $\therefore$ SBS' is a straight line, S being the focus of the tube.

Evidently SS' = the sum of the perpendiculars from B on the directrices $=NN'$. Also $S'P=PN$; $\therefore$ $SP=PN'$. Also the tangent at P is equally inclined to SP and PN'. Hence if we draw a parabola, focus S and directrix the horizontal through N', it will touch the parabolic path at P. This fixed parabola is $\therefore$ the envelope of all the parabolic paths.

738. [A. 1. b.] *Prove that*

$$\frac{{}^{2n-1}S_n}{(2n)!}-\frac{{}^{2n-2}S_n}{1!(2n-1)!}+\frac{{}^{2n-3}S_n}{2!(2n-2)!}-\dots+\frac{(-1)^{n-1}\,{}^nS_n}{(n-1)!(n+1)!}=\frac{1}{2^n.n!},$$

where ${}^nS_r$ *denotes the sum of the products r together of the first n natural numbers.*

We have, by the Binomial Theorem, as long as the series is convergent,

$$(1-xy)^{-\frac{1}{x}}=1+\sum_1^\infty\frac{(1+x)(1+2x)\dots(1+\overline{r-1}x)}{r!}y^r,$$

and the coefficient of x^p in the product in the numerator is ${}^{r-1}S_p$.

Hence evidently the given series is the coefficient of x^ny^{2n} in the product of the series written above, and the series

$$1-\frac{y}{1!}+\frac{y^2}{2!}-\dots,$$

i.e. the coefficient of x^ny^{2n} in $e^{-y}(1-xy)^{-\frac{1}{x}}=e^{-y-\frac{1}{x}\log(1-xy)}$.

Now $$-\frac{1}{x}\log(1-xy)=y+\frac{xy^2}{2}+\frac{x^2y^3}{3}+\dots.$$

Hence we want the coefficient of x^ny^{2n} in $e^{\frac{xy^2}{2}+\frac{x^2y^3}{3}+\dots}$.

But in this expansion x^ny^{2n} only occurs once, viz. in the term

$$\left(\frac{xy^2}{2}+\frac{x^2y^3}{3}+\dots\right)^n\Big/n!,$$

and its coefficient is $$\frac{1}{2^n.n!}.$$

739. [K. 11. c; K. 20. e.] *A is a vertex of a regular polygon of n sides inscribed in a circle whose centre is O and radius a. P is the middle point of OA. Prove that the sum of the fourth powers of the distances of the vertices of the polygon from P is* $\frac{33n}{16}.a^4$.

Let the vertices counting from A be $A_1, A_2, \dots, A_{n-1}$. Then

$$PA_r^2=a^2+\left(\frac{a}{2}\right)^2-2a\left(\frac{a}{2}\right)\cos\frac{2r\pi}{n}=a^2\left(\frac{5}{4}-\cos\frac{2r\pi}{n}\right).$$

Hence $$\Sigma PA_r^4=a^4\left(\frac{25}{16}n-\frac{5}{2}\sum_1^n\cos\frac{2r\pi}{n}+\sum_1^n\cos^2\frac{2r\pi}{n}\right).$$

Now $\sum_1^n \cos^2\frac{2r\pi}{n} = \frac{1}{2}\sum_1^n\left(\cos\frac{4r\pi}{n}+1\right) = \frac{1}{2}n$, since $\sum_1^n \cos\frac{4r\pi}{n} = 0$.

Also $\sum_1^n \cos\frac{2r\pi}{n} = 0$.

Hence $$\Sigma PA_r^4 = a^4\left(\frac{25}{16}n + \frac{1}{2}n\right) = \frac{33n}{16}\cdot a^4.$$

740. [L^1. 4. a.] *Perpendiculars SM, $S'N$ are drawn to the focal distances SP, $S'P$ of any point P on an ellipse to meet the tangent at P in M and N. Prove that if the eccentricity of the ellipse is not less than $\frac{1}{\sqrt{2}}$, the minimum value of the rectangle $PM.PN$ is $4b^2$, but that otherwise it is $\frac{b^2}{e^2(1-e^2)}$.*

Let α be the eccentric angle of $P(x, y)$ and θ the angle the tangent at P makes with the major axis. Then since M and N are on the directrices,

$$\therefore\ PM.PN = \left(\frac{a^2}{e^2} - x^2\right)\sec^2\theta.$$

Now $\tan\theta = -\frac{b}{a}\cot\alpha$; $\therefore\ \sec^2\theta = \frac{1-e^2\cos^2\alpha}{\sin^2\alpha}$;

$$\therefore\ PM.PN = a^2\left(\frac{1}{e^2} - \cos^2\alpha\right)\cdot\frac{1-e^2\cos^2\alpha}{\sin^2\alpha} = \frac{a^2}{e^2}\left(\frac{1-e^2\cos^2\alpha}{\sin\alpha}\right)^2.$$

Now $\frac{1-e^2\cos^2\alpha}{\sin\alpha} = \frac{1-e^2}{\sin\alpha} + e^2\sin\alpha$, and the minimum value of this is $2e\sqrt{1-e^2}$ occurring when $\frac{1-e^2}{\sin\alpha} = e^2\sin\alpha$, *i.e.* $\frac{1-e^2}{e^2} = \sin^2\alpha$. This only gives a possible value of α when $1-e^2 \ngtr e^2$, *i.e.* $e \nless \frac{1}{\sqrt{2}}$, and the required minimum is

$$\frac{a^2}{e^2}\cdot 4e^2(1-e^2) = 4b^2.$$

Otherwise there is no minimum, but the quantity decreases as α increases, and has its least value when $\alpha = \frac{\pi}{2}$, that value being $\frac{a^2}{e^2}$.

741. [L^1. 4. a.] *Prove that the area of the triangle formed by two tangents to $S=0$ and their chord is*

$$\frac{S^{\frac{3}{2}}\sqrt{-\Delta}}{CS-\Delta},$$

where $S \equiv ax^2+2hxy+by^2+2gx+2fy+c$, and C and Δ have their usual meanings.

If we change to the principal axes the equation takes the form

$$\alpha X^2 + \beta Y^2 + c' = 0 \quad\text{where}\quad c' = \frac{\Delta}{ab-h^2} = \frac{\Delta}{C}.$$

We now find (as in Solution 734) that the square of the length of the chord of contact of tangents from (X, Y) is

$$\frac{-4c'(\alpha^2X^2+\beta^2Y^2)(\alpha X^2+\beta Y^2+c')}{\alpha\beta(\alpha X^2+\beta Y^2)^2},$$

while the perpendicular from (X, Y) on the chord is $\frac{\alpha X^2+\beta Y^2+c'}{\sqrt{\alpha^2X^2+\beta^2Y^2}}$.

Hence the area of the triangle is

$$\frac{(aX^2+\beta Y^2+c')^{\frac{3}{2}}}{aX^2+\beta Y^2}\cdot\sqrt{-\frac{c'}{a\beta}}.$$

Now $a\beta=C$, and hence transferring to the old axes, this is

$$\frac{S^{\frac{3}{2}}}{S-c'}\,\frac{\sqrt{-\Delta}}{C}=\frac{S^{\frac{3}{2}}.\sqrt{-\Delta}}{CS-\Delta}.$$

742. [R. 9. a.] *A cylinder of weight W' and radius r stands on a rough horizontal table, at a distance c from an edge; over the cylinder is slipped a smooth circular ring of weight w. A rod of length $2l$ and weight W is smoothly jointed to the ring and rests projecting over the edge, while the ring touches the cylinder at two points. Neglecting the friction between the rod and table, prove that*

$$\cos^3\theta=(c/l)(1+w/W),$$

where θ is the inclination of the rod to the horizon. Prove also that in order that the cylinder may not upset $(W+w)c\tan^2\theta-wr$ must be less than rW'.

Let R be the reaction between the rod and the table, R' that between the ring and the cylinder, P the point of attachment of the rod to the ring. Then resolving vertically for the ring and rod,

$$W+w=R\cos\theta,$$

and horizontally, $$R'=R\sin\theta=(W+w)\tan\theta.$$

Taking moments about P for the rod,

$$W.l\cos\theta=R.c\sec\theta=(W+w)c\sec^2\theta;\quad\therefore\ \cos^3\theta=\frac{W+w}{W}\cdot\frac{c}{l}.$$

Again, let x be the vertical distance between the two points of contact of the ring Then, taking moments about P for the ring, $R'x=wr$. The condition that the cylinder should not upset is

$$R'(c\tan\theta-x)<W'r,$$

$$i.e.\ (W+w)c\tan^2\theta-wr<W'r.$$

743. [R. 7. b. γ.] *Two equal inelastic particles A and B connected by a tight inextensible string rest on a table, and A is constrained to move in a straight horizontal groove inclined at an angle 45° to the string. If a third equal particle C is projected perpendicular to the groove with velocity V so as to strike B directly, shew that A starts off with velocity $\frac{1}{5}V$.*

Let m be the mass of each particle, I the impulse between C and B, T the impulsive tension of the string, U the initial velocity of A, V_1 that of C after impact, u and v those of B along and perpendicular to the groove after impact.

Then for the motion of C we have

$$I=m(V-V_1);\ \text{..............................(i)}$$

for that of B along and perpendicular to CB,

$$I-T\cos 45^\circ=mv,\ \text{..............................(ii)}$$

$$T\cos 45^\circ=mu,\ \text{..............................(iii)}$$

and for that of A along the groove,

$$T\cos 45^\circ = mU. \qquad\text{(iv)}$$

Also, since the string is inextensible,

$$(v-u)\cos 45^\circ = U\cos 45^\circ, \qquad\text{(v)}$$

and since B and C are inelastic, $\quad v = V_1 \qquad$ (vi)

From (iii) and (iv), $u = U$, and thence from (v), $v = 2u$.

Hence, by (ii) and (iii), $\quad I = m(v+u) = 3mu$,

and by (i) and (vi), $\quad I = m(V-2u)$;

$$\therefore\ 3u = V - 2u, \text{ i.e. } u = \tfrac{1}{5}V.$$

744. [K. 11. a.] *Prove that a circle which touches a given circle and cuts a given straight line at a given angle also touches a second circle, the straight line being the radical axis of the two circles.*

Invert so that the given line and the given circle become two concentric circles. Then the inverse of the variable circle is a circle touching one of these concentric circles and cutting the other at a constant angle. It is evident, then, that it always touches a third concentric circle, the inverse of which is a circle coaxal with the inverses of the other two, *i.e.* such that the given line is the radical axis of it and the other fixed circle.

745. [A. 1. b. c.] *Given that for all integral values of n*

$$u_{n+1} = u_0u_n + u_1u_{n-1} + u_2u_{n-2} + \ldots + u_nu_0$$

and $u_0 = 1,\ u_1 = 1$, *shew that*

$$u_n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\cdot\frac{2^{2n}}{n+1}.$$

Let $\quad S = u_0 + u_1x + u_2x^2 + \ldots + u_nx^n + \ldots,$

Then evidently, from the datum,

$$S^2 = u_0^2 + u_2x + u_3x^2 + \ldots + u_{n+1}x^n + \ldots$$

$$\text{i.e. } xS^2 = S - 1, \quad (\text{since } u_0 = 1,\ u_1 = 1),$$

provided the series S is convergent.

Hence, solving $\quad S = \dfrac{1 \pm \sqrt{1-4x}}{2x}$,

and since S only contains positive powers of x,

$$\therefore\ S = \frac{1}{2x}\{1 - (1-4x)^{\frac{1}{2}}\}.$$

Hence $\quad u_n = \text{coefficient of } x^{n+1} \text{ in } -\tfrac{1}{2}(1-4x)^{\frac{1}{2}}$

$$= -\tfrac{1}{2}\cdot\frac{\frac{1}{2}(\frac{1}{2}-1)\ldots(\frac{1}{2}-n)}{(n+1)!}(-4)^{n+1}$$

$$= \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\cdot\frac{4^n}{n+1}.$$

746. [D. 2. b.] *If* $a_1, a_2, a_3 \ldots$ *be all positive and* $a_1+a_2+a_3+\ldots$ *divergent, then*

$$\frac{a_1}{a_1+1}+\frac{a_2}{(a_1+1)(a_2+1)}+\frac{a_3}{(a_1+1)(a_2+1)(a_3+1)}+\ldots$$

is convergent and equal to unity.

The n^{th} term is

$$\frac{1}{(a_1+1)\ldots(a_{n-1}+1)}-\frac{1}{(a_1+1)\ldots(a_n+1)}.$$

Hence the sum of n terms is

$$1-\frac{1}{(a_1+1)(a_2+1)\ldots(a_n+1)}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Now since the a's are all positive;

$$\therefore \ (a_1+1)(a_2+1)\ldots(a_n+1)>1+\sum_1^n a_r$$

$$\text{i.e. } \frac{1}{(a_1+1)(a_2+1)\ldots(a_n+1)}<\frac{1}{1+\sum_1^n a_r}$$

But since the series $a_1+a_2+a_3+\ldots$ is divergent; $\therefore\ 1+\sum_1^n a_r$ can be made as great as we please by sufficiently increasing n, *i.e.* the sum (i) can be made to differ from unity by a quantity as small as we please, provided n is taken sufficiently great. Hence the original series is convergent and equal to unity.

747. [D. 2. b. α.] *If* $a_k=(-1)^{k-1}\cdot\dfrac{2(n!)^2}{k(n+k)!(n-k)!}$

shew that $a_1 \sin x+a_2 \sin 2x+\ldots+a_n \sin nx=x.$

By the ordinary rule of partial fractions,

$$\frac{1}{2z(z-1^2)(z-2^2)\ldots(z-n^2)}=\sum_{k=1}^{k=n}\frac{(-1)^{n-k}}{(n+k)!(n-k)!}\cdot\frac{1}{z-k^2}+\frac{(-1)^n}{2(n!)^2}\cdot\frac{1}{z},$$

or, putting $z=\dfrac{1}{y}$, we have

$$\frac{y^n}{2(1-1^2y)(1-2^2y)\ldots(1-n^2y)}=\sum_{k=1}^{k=n}\frac{(-1)^{n-k}}{(n+k)!(n-k)!}\cdot\frac{1}{1-k^2y}+\frac{(-1)^n}{2(n!)^2}.$$

Hence, expanding in powers of y, we see that

$$\sum_{k=1}^{k=n}\frac{(-1)^{n-k}}{(n+k)!(n-k)!}k^{2r}=0, \quad (r<n)$$

and

$$\sum_{k=1}^{k=n}\frac{(-1)^{n-k}}{(n+k)!(n-k)!}+\frac{(-1)^n}{2(n!)^2}=0,$$

$$\text{i.e. } \sum_{k=1}^{k=n}a_k k^{2r+1}=0, \text{ and } \sum_{k=1}^{k=n}a_k k=1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

But $\sin\theta$ lies between

$$\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\ldots+(-1)^{r-1}\cdot\frac{\theta^{2r-1}}{(2r-1)!} \text{ and } \theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\ldots+(-1)^r\cdot\frac{\theta^{2r+1}}{(2r+1)!},$$

for a suitable value of r. Substituting these expressions in

$$a_1 \sin x+a_2 \sin 2x+\ldots+a_n \sin nx,$$

the result in each case is x by virtue of (i). Hence, the value of the series is x.

748. [L[1]. 19. 8.] *Through a point P on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *straight lines* PQR, $PQ'R'$ *are drawn parallel to the asymptotes of the confocal through* P, *meeting the major axis in* Q, Q' *and the minor axis in* R, R'. *Prove that* QR', $Q'R$ *intersect at a point on the normal at* P, *and that the locus of this point is*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{a^2-b^2}{a^2+b^2}\right)^2.$$

Let P be (X, Y) and the confocal $\frac{x^2}{a'^2}-\frac{y^2}{b'^2}=1$, so that $a'^2+b'^2=a^2-b^2$. Solving $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$, $\frac{X^2}{a'^2}-\frac{Y^2}{b'^2}=1$, and using this condition we find

$$\frac{X^2}{a^2a'^2}=\frac{Y^2}{b^2b'^2}=\frac{1}{a^2-b^2}=\frac{b'^2X^2+a'^2Y^2}{a'^2b'^2(a^2+b^2)}. \quad\text{......................(i)}$$

The lines through (X, Y) parallel to the asymptotes are

$$\frac{x-X}{a'}=\pm\frac{y-Y}{b'}.$$

Hence $$Q \text{ is } \left(X-\frac{a'}{b'}Y, 0\right),\quad R, \left(0, Y-\frac{b'}{a'}X\right),$$

$$Q', \left(X+\frac{a'}{b'}Y, 0\right),\quad R', \left(0, Y+\frac{b'}{a'}X\right).$$

Forming then the equations to QR', $Q'R$, they intersect where

$$x\left(X^2.\frac{b'}{a'}+Y^2.\frac{a'}{b'}\right)-a'b'X=0,\quad y\left(X^2.\frac{b'}{a'}+Y^2.\frac{a'}{b'}\right)+a'b'Y=0,$$

i.e. by (i) $$\frac{x}{X}=\frac{y}{-Y}=\frac{a^2-b^2}{a^2+b^2}.$$

These values satisfy $a^2xY-b^2yX=(a^2-b^2)XY$ and $\therefore$ this point lies on the normal at P. Also since $\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$, its locus is evidently

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{a^2-b^2}{a^2+b^2}\right)^2.$$

749. [L[1]. 17. e.] *A circle touches the conic* $\frac{l}{r}=1+e\cos\theta$ *and passes through the pole. Shew that the envelope of the chord joining the other two points of intersection is the conic*

$$\frac{l^2}{r^2}\left(\frac{1-e^2}{e^2}\right)^2+\frac{2l}{r}\cdot\frac{1-e^2}{e}\cos\theta+e^2\cos^2\theta=1.$$

If the circle and conic touch at α, the Cartesian equation of the circle must be of the form

$$(1-e^2)x^2+y^2+2elx-l^2$$
$$+\lambda[(e+\cos\alpha)x+\sin\alpha.y-l][(e+\cos\alpha)x-\sin\alpha.y-l']=0,$$

the necessary conditions being

$$1-e^2+\lambda(e+\cos\alpha)^2=1-\lambda\sin^2\alpha,\quad l=\lambda l',$$

whence $$\lambda=\frac{e^2}{1+2e\cos\alpha+e^2},$$

so that the polar equation to the chord of intersection is

$$\frac{l}{r} \cdot \frac{1+2e\cos\alpha+e^2}{e^2} = e\cos\theta + \cos(\theta+\alpha).$$

But the envelope of $L\cos\alpha + M\sin\alpha + N = 0$ is $L^2+M^2=N^2$. Hence here the envelope is

$$\left(\cos\theta - \frac{2l}{er}\right)^2 + \sin^2\theta = \left[e\cos\theta - \frac{l}{r}\left(1+\frac{1}{e^2}\right)\right]^2,$$

$$i.e.\quad \frac{l^2}{r^2}\left(1-\frac{1}{e^2}\right)^2 + \frac{2l}{r}\left(\frac{1}{e}-e\right)\cos\theta + e^2\cos^2\theta = 1.$$

750. [**R. 1. f.**] *In that system of pulleys in which each hangs by a separate string, there are three moveable pulleys, each of weight w, and a fixed pulley: round the latter passes the string from which the weight P, the motive power, hangs freely. When the weight W is ascending, the first and second strings (reckoning from the bottom) are joined by a new string, and just before the latter becomes tight the old strings are cut so as to release the central moveable pulley. Prove that the velocity of W is increased in the ratio*

$$(32P+9w+W):(16P+5w+W).$$

If the original velocity of the bottom pulley is v, those of the other pulleys are $2v$, $4v$, and that of P is $8v$. If the velocity of the bottom pulley after the jerk is v', that of the other pulley is $2v'$, and that of P is $4v'$. Hence, if T_1, T_2 are the impulsive tensions, we have

$$2T_1 = \frac{W+w}{g}(v'-v), \quad 2T_2 - T_1 = \frac{w}{g}(2v'-4v), \quad -T_2 = \frac{P}{g}(4v'-8v).$$

Multiplying these by 1, 2, 4 and adding, we get

$$(W+w)(v'-v) + w(4v'-8v) + P(16v'-32v) = 0,$$

$$i.e.\quad (W+5w+16P)v' = (W+9w+32P)v.$$

751. [**R. 7. b. γ.**] *A particle P of mass M rests in equilibrium on a smooth horizontal table, being attached to three particles of masses m_1, m_2, m_3 by fine strings which pass over smooth pulleys A, B, C at the edge of the table. Prove that if the string which supports m_3 be cut, the particle will begin to move in a direction making with CP an angle*

$$tan^{-1}\frac{(m_1-m_2)\{(m_1+m_2)^2-m_3^2\}\Delta}{4Mm_1m_2m_3^2+(m_1+m_2)\Delta^2}$$

where $$\Delta^2 = 2\Sigma m_2^2m_3^2 - \Sigma m_1^4.$$

Let α, β, γ be the angles between the strings in equilibrium, so that

$$\frac{\sin\alpha}{m_1} = \frac{\sin\beta}{m_2} = \frac{\sin\gamma}{m_3} = k, \text{ suppose.}$$

Now $\sin\alpha$, $\sin\beta$, $\sin\gamma$ are connected by the identical relation

$$2\Sigma\sin^2\beta\sin^2\gamma - \Sigma\sin^4\alpha \equiv 4\sin^2\alpha\sin^2\beta\sin^2\gamma$$

(as is easily obtained from $1-\Sigma\cos^2\alpha + 2\cos\alpha\cos\beta\cos\gamma \equiv 0$).

Hence

$$k^4\Delta^2 = 4k^6m_1^2m_2^2m_3^2\,; \quad \therefore\ k = \frac{\Delta}{2m_1m_2m_3}, \quad i.e.\ \sin\alpha = \frac{\Delta}{2m_2m_3}, \text{ etc.}$$

Now let the accelerations of M parallel and perpendicular to CP be f and f' and let T, T' be the tensions of the strings.

Then $$T'\sin\alpha - T\sin\beta = Mf'. \quad\text{.................................(i)}$$

Also the accelerations of m_1 and m_2 are the same as those of M along the respective strings. Hence

$$m_1g - T = m_1(-f\cos\beta - f'\sin\beta), \quad\text{..........................(ii)}$$

$$m_2g - T' = m_2(-f\cos\alpha + f'\sin\alpha). \quad\text{..........................(iii)}$$

From (i), (ii) and (iii) we have

$$m_1\sin\beta(-f\cos\beta - f'\sin\beta) - m_2\sin\alpha(-f\cos\alpha + f'\sin\alpha) = Mf'.$$

Hence $$\frac{f'}{f} = \frac{m_2\sin\alpha\cos\alpha - m_1\sin\beta\cos\beta}{M + m_1\sin^2\beta + m_2\cos^2\alpha}.$$

Now $$\cos\alpha = \frac{m_2^2 + m_3^2 - m_1^2}{2m_2m_3}, \quad \cos\beta = \frac{m_3^2 + m_1^2 - m_2^2}{2m_3m_1}.$$

Substituting these and the values of $\sin\alpha$, $\sin\beta$, and reducing, we obtain the required expression.

752. [L[1]. 11. b.] *If the normal at P to a rectangular hyperbola meet the curve in Q, prove that*

$$PQ^2 = 3CP^2 + CQ^2$$

where C is the centre.

P and Q must lie on opposite branches. Let V be the middle point of PQ, $-d^2$ the square of the imaginary semi-diameter in direction CV. Draw the diameter CP' parallel to PQ. Then since CP', CV are the directions of conjugate diameters; $\therefore$ $CP' = d$. Also CP' is perpendicular to the diameter conjugate to CP, and $\therefore$ $CP = d$.

Now $$PV^2 = CV^2 + d^2 = CV^2 + CP^2.$$

But $$CP^2 + CQ^2 = 2CV^2 + 2PV^2 = 2PV^2 - 2CP^2 + 2PV^2$$
$$= PQ^2 - 2CP^2;$$
$$\therefore\ 3CP^2 + CQ^2 = PQ^2.$$

753. [A. 3. g.] *Prove that the conditions that u can be so chosen that the equations*

$$(a+u)x^2 + 2(b+u)x + (c+u) = 0,$$
$$(a'+u)x^2 + 2(b'+u)x + (c'+u) = 0$$

may both have real roots are either

$$(a+c-2b)(a'+c'-2b') > 0$$

or $$(a+c-2b) > 0 > (a'+c'-2b'),$$

together with $$(b^2-ac)/(a+c-2b) - (b'^2-a'c')/(a'+c'-2b') > 0,$$

or similar conditions with the accented and unaccented letters interchanged.

Let 4Δ, $4\Delta'$ be the discriminants, so that

$$\Delta = (b+u)^2 - (a+u)(c+u)$$
$$= b^2 - ac - (a+c-2b)u$$
$$= (a+c-2b)\left[\frac{b^2-ac}{a+c-2b} - u\right],$$

and similarly for Δ'.

The necessary condition is that Δ and Δ' should be both positive. Now if $a+c-2b$ and $a'+c'-2b'$ have the same sign, it is evident that u can be chosen so that the quantities in the square brackets have both the same sign, positive or negative as may be required to make Δ, Δ' both positive.

Again $$\frac{\Delta}{a+c-2b}-\frac{\Delta'}{a'+c'-2b'}=\frac{b^2-ac}{a+c-2b}-\frac{b'^2-a'c'}{a'+c'-2b'}.$$

Now if $a+c-2b>0>a'+c'-2b'$, and Δ, Δ' are both positive, then the left side is positive and $\therefore$ the right side is positive, and conversely. With the other set of conditions both sides are negative.

754. [I. 2. c.] *Prove that*
$$\frac{(2m-1)!}{m!(m-1)!}$$
is an even number, except when m is a power of 2.

Any odd number can be expressed in the form
$$2^{a_1}+2^{a_2}+\dots+2^{a_{k-1}}+2^{a_k},$$
where $a_1>a_2>a_3\dots$, and $a_k=0$.

If this is $2m-1$, then $m=2^{a_1-1}+2^{a_2-1}+\dots+2^{a_{k-1}-1}$.

The highest power of 2 contained in $(2m-1)!$ is
$$I\left(\frac{2m-1}{2}\right)+I\left(\frac{2m-1}{2^2}\right)+\dots$$
$$=(2^{a_1-1}+2^{a_1-2}+\dots+1)$$
$$+(2^{a_2-1}+2^{a_2-2}+\dots+1)$$
$$+\dots+(2^{a_k-1}+2^{a_k-2}+\dots+1)$$
$$=(2^{a_1}-1)+(2^{a_2}-1)+\dots+(2^{a_k}-1)=(2m-1)-k.$$

Similarly the highest powers contained in $m!$ and $(m-1)!$ are $m-k$ and $(m-1)-(k-1)$, *i.e.* in each case $(m-k)$.

Hence the quotient has a factor 2^{k-1}, and is $\therefore$ always even, since $k>1$.

If $m=2^a$, the highest powers of 2 contained in $m!$, $(2m-1)!$ and $(m-1)!$ are $m-1$, $(2m-1)-a$, and $(m-1)-(a-1)$ respectively, so that the quotient contains no power of 2.

E. M. Radford.

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